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Stability results for impulsive functional differential equations with infinite delays

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Abstract

This paper studies the stability problems for a class of impulsive functional differential equations with infinite delays of the form

$$x'(t) = F(t, x(\cdot)), \quad t > t^*,$$

$$x(t_k) = J_k(x(t_k^-)), \quad k = 1, 2, \ldots .$$

By using the Liapunov functions and Razumikhin technique, some new Razumikhin-type theorems on stability are obtained.

MSC: 34D20; 34K20

Keywords: Stability; Impulsive differential equation; Razumikhin technique

1. Introduction and preliminaries

Systems of differential equations with impulsive effect provide mathematical models for many phenomena and processes in the field of natural sciences and technology [4,11]. The stability theory of impulsive differential equations goes back to the work of Mil'man and Myshkis [12]. In the last few decades, the stability theory of impulsive differential equations marked a rapid development and most of the research focused on impulsive ordinary differential equations. See, for example, [4,11] and the references cited therein. Now, there also exists a well-developed stability theory of functional differential equations. However, not much has been developed in the direction of the stability theory of impulsive functional differential equations. In the few publications dedicated to

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this subject, the earlier works were done by Anokhin [1] and Gopalsamy and Zhang [7]. Recently, stability problems on some linear impulsive delay differential equations are systematically investigated in several papers. See, for example, [2,3,19,20]. In particular, in [13,15–18], the well-known Liapunov's second method applied to impulsive functional differential equations in more general form was suggested for stability analysis, and some interesting results were obtained for such equations with finite delays. It is well known that Liapunov's second method applied to infinite delay equations is more complicated than that applied to finite delay equations. The common and main difficulty is that the interval $(-\infty, t_0]$ is not compact, and the images of a solution map of closed and bounded sets in $C((-\infty,0],\mathbb{R}^n)$ space may not be compact. Same situation arises in $\mathrm{PC}((-\infty,0],\mathbb{R}^n)$ space for impulsive functional differential equations with infinite delays. Therefore, it is interesting to study the stability problem than related impulsive functional differential equations with infinite delays.

In this paper, we consider the system of functional differential equations with infinite delays of the form

$$x'(t) = F(t, x(\cdot)), \quad t > t^* \tag{1.1}$$

under the impulsive perturbed conditions

$$x(t_k) = J_k(x(t_k^-)), \quad k = 1, 2, \dots . \tag{1.2}$$

We are concerned with the application of the concept of Liapunov–Razumikhin functions to the determination of sufficient conditions for the stability of the systems (1.1) and (1.2). It is well known that Liapunov–Razumikhin function methods have been widely used in the treatment of the stability of various functional differential equations without impulses (cf. [5,6,8–10,14]). A manifest advantage of this method is that it does not demand the knowledge of solutions and therefore has great power in applications. Such a method applied to impulsive functional differential equations with finite delay can be found in [16,18].

Let $R=(-\infty,\infty)$, $R^+=[0,\infty)$. For $x \in \mathbb{R}^n$, $|\cdot|$ denotes the Euclidean norm for x. For $t \geqslant t^* > a \geqslant -\infty$, $F(t, x(s); a \leqslant s \leqslant t)$ or $F(t, x(\cdot))$ is a Volterra-type functional (cf. [6]), its values are in $\mathbb{R}^n$ and are determined by $t \geqslant t^*$ and the values of $x(s)$ for $[a,t]$. In (1.1), $x'(t)$ denotes the right-hand derivative of x at t. In (1.2), $t^* < t_k < t_{k+1}$ with $t_k \to \infty$ as $k \to \infty$, $J_k : \mathbb{R}^n \to \mathbb{R}^n$, and $x(t_k^-) = \lim_{t \to t_k - 0} x(t)$.

Let $I \subset R$ be any interval. Define $\mathrm{PC}(I, \mathbb{R}^n) = \{x : I \to \mathbb{R}^n,\ x$ is continuous everywhere except at the points $t = t_k \in I$ and $x(t_k^-)$ and $x(t_k^+) = \lim_{t \to t_k + 0} x(t)$ exist with $x(t_k^+) = x(t_k)\}$. For any $t \geqslant t^*$, $\mathrm{PC}([a,t], \mathbb{R}^n)$ will be written as $\mathrm{PC}(t)$. Define $\mathrm{PC}B(t) = \{x \in \mathrm{PC}(t)$: x is bounded$\}$. For any $\phi \in \mathrm{PC}B(t)$, the norm of ϕ is defined by

$$||\phi|| = ||\phi||^{[a,t]} = \sup_{a \leqslant s \leqslant t} |\phi(s)|.$$

For given $\sigma \geqslant t^*$ and $\phi \in \mathrm{PC}B(\sigma)$, with Eqs. (1.1) and (1.2), one associates an initial condition of the form

$$x(t) = \phi(t), \quad a \leqslant t \leqslant \sigma. \tag{1.3}$$

Definition 1.1. A function $x(t)$ is called a solution corresponding to σ of the initial value problem (1.1)–(1.3) if $x : [a,b) \to \mathbb{R}^n$ (for some $t^* < b \leqslant \infty$) is continuous for $t \in [a,b) \setminus \{t_k,\ k = 1,2,\dots\}$, $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^+) = x(t_k)$, and satisfies (1.1)–(1.3).

Under the following hypotheses (H_1)–(H_4), the initial value problem (1.1)–(1.3) exists with a unique solution which will be written in the form $x(t,\sigma,\phi)$. See [15].

(H_1) F is continuous on $[t_{k-1},t_k)\times PC(t)$ for $k=1,2,\ldots,$ where $t_0=t^*$. For all $\varphi\in PC(t)$ and $k=1,2,\ldots,$ the limits $\lim_{(t,\phi)\to(t_k^-,\varphi)}F(t,\phi)=F(t_k^-,\varphi)$ exist.

(H_2) F is locally Lipschitzian in ϕ in each compact set in $PCB(t)$. More precisely, for every $c\in[a,b)$ and every compact set $G\subset PCB(t)$ there exists a constant $L=L(c,G)$ such that

$$|F(t,\varphi(\cdot))-F(t,\psi(\cdot))|\leqslant L||\varphi-\psi||^{[a,t]}$$

whenever $t\in[a,c]$ and $\varphi,\psi\in G$.

(H_3) For each $k=1,2,\ldots,J_k(x)\in C(\mathbb{R}^n,\mathbb{R}^n)$.

(H_4) For any $x(t)\in PC([a,\infty),\mathbb{R}^n),F(t,x(\cdot))\in PC([t^*,\infty),\mathbb{R}^n)$.

For any $t\geqslant t^*,\ h>0$, Let

$$PCB_h(t)=\{\phi\in PCB(t)\colon ||\phi||<h\}.$$

In this paper, we assume that $F(t,0)\equiv 0,J_k(0)\equiv 0$ so that $x(t)\equiv 0$ is a solution of (1.1) and (1.2), which we call the zero solution. Also, throughout the paper, we will assume that $b=\infty$. More precisely, we will only consider the solutions $x(t,\sigma,\varphi)$ of Eqs. (1.1) and (1.2) which can continue to ∞ from the right of σ.

Definition 1.2. The zero solution of (1.1) and (1.2) is said to be

(S_1) Uniformly stable (US for short), if for any $\sigma\geqslant t^*$ and $\varepsilon>0$, there is a $\delta=\delta(\varepsilon)>0$ such that $\varphi\in PCB_\delta(\sigma)$ implies $|x(t,\sigma,\varphi)|\leqslant\varepsilon$ for $t\geqslant\sigma$.

(S_2) Uniformly asymptotically stable (UAS), if it is US, and there exists a $\delta>0$ such that for any $\varepsilon>0$ there is a $T=T(\varepsilon)>0$ such that $\sigma\geqslant t^*$ and $\varphi\in PCB_\delta(\sigma)$ imply $|x(t,\sigma,\varphi)|\leqslant\varepsilon$ for $t\geqslant\sigma+T$.

Definition 1.3. A function $V(t,x):[a,\infty)\times\mathbb{R}^n\to R^+$ belongs to class v_0 if

(A_1) V is continuous on each of the sets $[t_{k-1},t_k)\times\mathbb{R}^n$ and for all $x\in\mathbb{R}^n$ and $k\in Z^+$, the limits $\lim_{(t,y)\to(t_k^-,x)}V(t,y)=V(t_k^-,x)$ exist.

(A_2) V is locally Lipschitzian in x and $V(t,0)\equiv 0$.

Let $V\in v_0$, for any $(t,x)\in[t_{k-1},t_k)\times\mathbb{R}^n$, the right-hand derivative $V'(t,x)$ along the solution $x(t)$ of (1.1) and (1.2) is defined by

$$V'(t,x(t))=\limsup_{h\to 0^+}\frac{V(t+h,x(t+h))-V(t,x(t))}{h}.$$

We say a function $W:[0,\infty)\to[0,\infty)$ belongs to class $\mathscr{R}$ if W is continuous and strictly increasing and satisfies $W(0)=0$.

2. Main results

Theorem 2.1. *Let $W_i \in \mathscr{R}$ $(i=1,2,3)$, $V(t,x) \in v_0$, and $q \in C(R^+, R^+)$ such that $q(s)$ is nonincreasing, $q(s) > 0$ for $s > 0$. Assume that the following conditions hold*:

(i) $W_1(|x|) \leqslant V(t,x) \leqslant W_2(|x|)$,

(ii) *for each $k \in Z^+$ and all $x \in \mathbb{R}^n$*,

$$V(t_k, J_k(x)) \leqslant (1 + b_k)V(t_k^-, x),$$

where $b_k \geqslant 0$ with $\sum_{k=1}^{\infty} b_k < \infty$,

(iii) *for some $\beta_0 > 0$, any α, $0 < M\alpha \leqslant \beta_0$ and any $\lambda > 0$, there exists a number $\eta = \eta(\alpha, \beta_0, \lambda) > 0$ such that when $V(t,x(t)) \geqslant \alpha$, $\sup V(s,x(s)) \leqslant \beta_0$ and $V(s,x(s)) \leqslant MV(t,x(t)) + \eta$ for $\max\{a, t - q(V(t,x(t)))\} \leqslant s \leqslant t$, we have*

$$V'(t,x(t)) \leqslant -W_3(|x(t)|) + \lambda,$$

where $M = \prod_{k=1}^{\infty}(1 + b_k)$, $x(t) = x(t,\sigma,\varphi)$ is the solution of (1.1) and (1.2) and $\sigma \geqslant t^*$, $\varphi \in \mathrm{PC}B(\sigma)$.

Then the zero solution of (1.1) and (1.2) is uniformly asymptotically stable.

Proof. For a given $\varepsilon > 0$ we suppose $W_1(\varepsilon) < \beta_0$ and choose a positive number $\delta = \delta(\varepsilon)$ such that $MW_2(\delta) \leqslant W_1(\varepsilon)$. Let $\sigma \geqslant t^*$, $\varphi \in \mathrm{PC}B_\delta(\sigma)$ and $x(t) = x(t,\sigma,\varphi)$ be the solution of (1.1) and (1.2). Set $V(t) = V(t,x(t))$, and let $\sigma \in [t_{m-1}, t_m)$ for some $m \in Z^+$, where $t_0 = t^*$. Then

$$W_1(|x(t)|) \leqslant V(t) \leqslant W_2(\delta) \leqslant M^{-1}W_1(\varepsilon), \quad a \leqslant t \leqslant \sigma.$$

We claim that

$$V(t) < M^{-1}W_1(\varepsilon), \quad \sigma \leqslant t < t_m. \tag{2.1}$$

Suppose there exists some $\bar{t} \in (\sigma, t_m)$ such that

$$V(\bar{t}) = \frac{1}{M}W_1(\varepsilon) \geqslant W_2(\delta), \tag{2.2}$$

$$V(t) \leqslant V(\bar{t}), \quad t \in (\sigma, \bar{t}) \tag{2.3}$$

then $\delta \leqslant |x(\bar{t})| \leqslant \varepsilon$, $V'(\bar{t}) \geqslant 0$ and $V(s) \leqslant V(\bar{t})$ for $a \leqslant s \leqslant \bar{t}$.

Thus, let λ such that $0 < \lambda < \inf_{\delta \leqslant s \leqslant \varepsilon} W_3(s)$, $\alpha = W_2(\delta) \leqslant \beta_0$. One has, $V(\bar{t}) > \alpha$, $\sup V(s) \leqslant \beta_0$, $V(s) \leqslant MV(\bar{t}) + \eta$, for any $\eta > 0$ and $\max\{a, \bar{t} - q(V(\bar{t}, x(\bar{t})))\} \leqslant s \leqslant \bar{t}$. By assumption (iii),

$$V'(\bar{t}) \leqslant -W_3(|x(\bar{t}|) + \lambda < 0.$$

This is a contradiction and so (2.1) holds. From (2.1) and assumption (ii) one has

$$V(t_m) = V(t_m, J_m(x(t_m^-))) \leqslant (1 + b_m)V(t_m^-) \leqslant M^{-1}(1 + b_m)W_1(\varepsilon).$$

Similarly, one can prove that

$$V(t) \leqslant M^{-1}(1 + b_m)W_1(\varepsilon), \quad t_m \leqslant t < t_{m+1},$$

$$V(t_{m+1}) \leqslant M^{-1}(1 + b_m)(1 + b_{m+1})W_1(\varepsilon)$$

and by induction, we can prove in general that for $i = 0, 1, \dots,$

$$V(t) \leqslant M^{-1}(1 + b_m) \cdots (1 + b_{m+i})W_1(\varepsilon), \quad t_{m+i} \leqslant t < t_{m+i+1},$$

$$V(t_{m+i+1}) \leqslant M^{-1}(1 + b_m) \cdots (1 + b_{m+i+1})W_1(\varepsilon).$$

Thus, one has

$$W_1(|x(t)|) \leqslant V(t) \leqslant W_1(\varepsilon), \quad t \geqslant \sigma$$

or $|x(t)| \leqslant \varepsilon$ for $t \geqslant \sigma$. This completes the proof of US.

Next we will show UAS. For a given $\varepsilon = \varepsilon_1 > 0$, we can choose a $\delta > 0$ such that $MW_2(\delta) = W_1(\varepsilon_1) \leqslant \beta_0$, in view of the proof of US, we know that $\varphi \in \mathrm{PC}B_\delta(\sigma)$ implies that

$$V(t) \leqslant W_1(\varepsilon_1), \quad |x(t)| \leqslant \varepsilon_1, \quad t \geqslant \sigma.$$

Now, let $\varepsilon > 0 (\varepsilon < \varepsilon_1)$, we will prove that there exists a $T = T(\varepsilon) > 0$ such that $\varphi \in \mathrm{PC}B_\delta(\sigma)$ implies that

$$|x(t)| \leqslant \varepsilon, \quad t \geqslant \sigma + T.$$

Set

$$\lambda = \tfrac{1}{2}\inf\{W_3(s)\colon W_2^{-1}(M^{-1}W_1(\varepsilon)) \leqslant s \leqslant \varepsilon_1\}, \quad \alpha = \frac{1}{M}W_1(\varepsilon),$$

$$h = \max\left\{\frac{W_1(\varepsilon_1)(1 + M^*)}{\lambda},\ q(M^{-1}W_1(\varepsilon))\right\},$$

where $M^* = \sum_{k=1}^{\infty} b_k$. From (iii), there exists a $\eta > 0$. Let N be the smallest positive integer such that $MW_2(\delta) \leqslant M^{-1}[W_1(\varepsilon) + N\eta]$. Let

$$\tau_i = \sigma + 2ih, \qquad i = 0, 1, \dots, N.$$

We will prove that

$$V(t) \leqslant W_1(\varepsilon) + (N - i)\eta, \quad t \geqslant \tau_i,\ i = 0, 1, \dots, N. \tag{2.4$_i$}$$

Clearly, (2.4_0) holds. Now, suppose (2.4_i) holds for some $0 \leqslant i < N$. We prove that

$$V(t) \leqslant W_1(\varepsilon) + (N - i - 1)\eta, \quad t \geqslant \tau_{i+1}, \quad i = 0, 1, \dots, N. \tag{2.4$_{i+1}$}$$

We first claim that there exists a $\bar{t} \in I_i = [\tau_i + h, \tau_{i+1}]$ such that

$$V(\bar{t}) \leqslant M^{-1}[W_1(\varepsilon) + (N - i - 1)\eta]. \tag{2.5}$$

Suppose for all $t \in I_i$,

$$V(t) > M^{-1}[W_1(\varepsilon) + (N - i - 1)\eta],$$

then for such t we have

$$M^{-1}W_1(\varepsilon) < V(t) \leqslant W_1(\varepsilon_1)$$

and so

$$W_2^{-1}(M^{-1}W_1(\varepsilon)) \leqslant |x(t)| \leqslant \varepsilon_1$$

$$V(s) \leqslant W_1(\varepsilon) + (N - i)\eta < MV(t) + \eta, \quad t - h \leqslant s \leqslant t.$$

In view of the definition of h and noting that q is nonincreasing, one has for $t \in I_i$, $V(t) > \alpha$, $\sup V(s) \leqslant W_1(\varepsilon_1) \leqslant \beta_0$, $V(s) \leqslant MV(t) + \eta$, for $\max\{a, t - q(V(t, x(t)))\} \leqslant s \leqslant t$. By assumption (iii), we have for $t \in I_i$,

$$V'(t) \leqslant -W_3(|x(t)|) + \lambda \leqslant -\lambda.$$

Thus, for $t \in I_i$,

$$\begin{aligned} V(t) &\leqslant V(\tau_i + h) - \lambda(t - \tau_i - h) + \sum_{\tau_i + h \leqslant t_k < t} [V(t_k) - V(t_k^-)] \\ &\leqslant W_1(\varepsilon_1) - \lambda(t - \tau_i - h) + \sum_{k=1}^{\infty} b_k V(t_k^-) \\ &\leqslant (1 + M^*) W_1(\varepsilon_1) - \lambda(t - \tau_i - h). \end{aligned}$$

Set $t = \tau_{i+1}$, we have

$$V(\tau_{i+1}) \leqslant (1 + M^*) W_1(\varepsilon_1) - \lambda \frac{(1 + M^* W_1(\varepsilon_1))}{\lambda} = 0.$$

It is a contradiction and so (2.5) holds for some $\bar{t} \in I_i$.

Let $l = \min\{k \in Z^+ \colon t_k > \bar{t}\}$. We claim that

$$V(t) \leqslant M^{-1}[W_1(\varepsilon) + (N - i - 1)\eta], \quad \bar{t} \leqslant t < t_l. \tag{2.6}$$

Otherwise, there exists a $\hat{t} \in (\bar{t}, t_l)$ such that

$$V(\hat{t}) > M^{-1}[W_1(\varepsilon) + (N - i - 1)\eta] \geqslant V(\bar{t})$$

which implies that there is a $\check{t} \in (\bar{t}, \hat{t}\,]$ such that

$$V'(\check{t}) > 0 \quad \text{and} \quad V(\check{t}) \geqslant M^{-1}[W_1(\varepsilon) + (N - i - 1)\eta].$$

On the other hand, for $\check{t} - h \leqslant s \leqslant \check{t}$,

$$MV(\check{t}) + \eta > W_1(\varepsilon) + (N - i)\eta \geqslant V(s)$$

and so

$$V(s) \leqslant MV(\check{t}) + \eta, \quad \max\{a, \check{t} - q(V(\check{t}))\} \leqslant s \leqslant \check{t}.$$

By assumption (iii),

$$V'(\check{t}) \leqslant -W_3(|x(\check{t})|) + \lambda \leqslant -\lambda \leqslant 0.$$

This is a contradiction and so (2.6) holds. From (2.6) and assumption (ii) we have

$$V(t_l) \leqslant (1 + b_l) V(t_l^-) \leqslant M^{-1}(1 + b_l)[W_1(\varepsilon) + (N - i - 1)\eta].$$

By induction, one can prove in general that

$$V(t) \leqslant M^{-1}(1 + b_l) \cdots (1 + b_{l+i})[W_1(\varepsilon) + (N - i - 1)\eta], \quad t_{l+i} \leqslant t < t_{l+i+1},$$

$$V(t_{l+i+1}) \leqslant M^{-1}(1 + b_l) \cdots (1 + b_{l+i+1})[W_1(\varepsilon) + (N - i - 1)\eta], \quad i = 0, 1, \ldots$$

thus,

$$V(t) \leqslant W_1(\varepsilon) + (N - i - 1)\eta, \quad t \geqslant \bar{t}.$$

Therefore, (2.4_{i+1}) holds. By induction, we know that (2.4_i) hold for all $i = 0, 1, \dots, N$. Thus, when $i = N$ we obtain

$$W_1(|x(t)|) \leqslant V(t) \leqslant W_1(\varepsilon), \quad t \geqslant \sigma + 2Nh.$$

Now, let $T = 2Nh$, then $|x(t)| \leqslant \varepsilon$ for $t \geqslant \sigma + T$. The proof is complete. □

Remark 2.2. From the proof of the uniform stability part in Theorem 2.1, one can easily see that if conditions (i) and (ii) are satisfied and condition (iii) is replaced

(iii′) for some $\beta_0 > 0$, any α, $0 < M\alpha \leqslant \beta_0$ and any $\lambda > 0$, there exists a number $\eta = \eta(\alpha, \beta_0, \lambda) > 0$ such that when $V(t, x(t)) \geqslant \alpha$, $\sup V(s, x(s)) \leqslant \beta_0$ and $V(s, x(s)) \leqslant MV(t, x(t)) + \eta$ for $a \leqslant s \leqslant t$, we have

$$V'(t, x(t)) \leqslant -W_3(|x(t)|) + \lambda,$$

where $M = \prod_{k=1}^{\infty}(1 + b_k)$, $x(t) = x(t, \sigma, \varphi)$ is the solution of (2.1) and (2.2).
Then the zero solution of (1.1) and (1.2) is uniformly stable.

As two convenient versions of Theorem 2.1, we have the following theorems.

Theorem 2.3. *Let $W_i \in \mathscr{R}(i=1,2,3)$, $V(t,x) \in v_0$, and $q \in C(R^+, R^+)$ such that $q(s)$ is nonincreasing, $q(s) > 0$ for $s > 0$. Assume that the following conditions hold*:

(i) $W_1(|x|) \leqslant V(t,x) \leqslant W_2(|x|)$,
(ii) *for each $k \in Z^+$ and all $x \in \mathbb{R}^n$*,

$$V(t_k, J_k(x)) \leqslant (1 + b_k)V(t_k^-, x),$$

where $b_k \geqslant 0$ with $\sum_{k=1}^{\infty} b_k < \infty$;
(iii) *for any $\sigma \geqslant t^*$ and $\varphi \in \mathrm{PC}B(\sigma)$*,

$$V'(t, x(t)) \leqslant -W_3(|x(t)|) \text{ if}$$

$$P(V(t, x(t)) > V(s, x(s)) \quad \max\{a, t - q(V(t, x(t)))\} \leqslant s \leqslant t,$$

where $P(s) \in C(R^+, R^+)$, $P(s) > Ms$ for $s > 0$, $M = \prod_{k=1}^{\infty}(1+b_k)$, $x(t) = x(t, \sigma, \varphi)$ is the solution of (1.1) *and* (1.2).

Then the zero solution of (1.1) *and* (1.2) *is uniformly asymptotically stable.*

Proof. For any $\alpha, \beta, 0 < M\alpha < \beta$ and any $\lambda > 0$, let $\eta = \inf_{\alpha \leqslant s \leqslant \beta}(P(s) - Ms)$. It is clear that $\eta > 0$. When $V(t, x(t)) \geqslant \alpha$, $\sup V(s, x(s)) \leqslant \beta$ and $V(s, x(s)) \leqslant MV(t, x(t)) + \eta$, for $\max\{a, t - q(V(t, x(t)))\} \leqslant s \leqslant t$ we have

$$V(s, x(s)) < MV(t, x(t)) + P(V(t, x(t))) - MV(t, x(t))$$

$$= P(V(t, x(t)) \text{ for } \max\{a, t - q(V(t, x(t)))\} \leqslant s \leqslant t.$$

By (iii) we get

$$V'(t, x(t)) \leqslant -W_3(|x(t)|) \leqslant -W_3(|x(t)|) + \lambda.$$

By the Theorem 2.1, the zero solution of (1.1) and (1.2) is uniformly asymptotically stable. □

Theorem 2.4. *Let $W_i \in \mathscr{R}(i=1,2,3), V(t,x) \in v_0$, and $q \in C(R^+,R^+)$ such that $q(s)$ is nonincreasing, $q(s) > 0$ for $s > 0$. Assume that the following conditions hold*:

(i) $W_1(|x|) \leqslant V(t,x) \leqslant W_2(|x|)$;

(ii) *for each $k \in Z^+$ and all $x \in \mathbb{R}^n$*,

$$V(t_k, J_k(x)) \leqslant (1+b_k)V(t_k^-, x),$$

where $b_k \geqslant 0$ with $\sum_{k=1}^{\infty} b_k < \infty$;

(iii) *there exists a function $G : R^+ \times R^+ \to R$ continuous and satisfying $G(v, Mv) = -W_3(v)$, such that for any solution of* (1.1) *and* (1.2)

$$V'(t,x(t)) \leqslant G\left(V(t,x(t)), \sup_{\max\{a,t-q(V(t,x(t)))\} \leqslant s \leqslant t} V(s,x(s))\right).$$

Then the zero solution of (1.1) *and* (1.2) *is uniformly asymptotically stable.*

Proof. For any $\alpha, \beta, 0 < M\alpha < \beta$ and any $\lambda > 0$, since G is uniformly continuous on $[\alpha, M\beta] \times [\alpha, M\beta]$, there is a $\delta > 0$ such that for any $u, v \in [\alpha, \beta]$, when $|u - Mv| \leqslant \eta$, one has

$$|G(v,u) - G(v,Mv)| \leqslant \lambda.$$

Let $V(t,x(t)) \geqslant \alpha$, $\sup V(s,x(s)) \leqslant \beta$ and $\sup V(s,x(s)) \leqslant MV(t,x(t)) + \eta$. Then, $|\sup V(s,x(s)) - MV(t,x(t))| \leqslant \eta$. For $\max\{a, t - q(V(t,x(t)))\} \leqslant s \leqslant t$. We have

$$\begin{aligned} V'(t,x(t)) &\leqslant -W_3(V(t,x(t)) + \Bigg|-G(V(t,x(t)), MV(t,x(t))) \\ &\qquad + G\left(V(t,x(t)), \sup_{\max\{a,t-q(V(t,x(t)))\} \leqslant s \leqslant t} V(s,x(s))\right)\Bigg| \\ &\leqslant -W_4(|x(t)|) + \lambda, \end{aligned}$$

where $W_4(z) = \inf_{W_1(z) \leqslant \theta \leqslant W_2(z)} W_3(\theta)$. By Theorem 2.1, the zero solution of (1.1) and (1.2) is uniformly asymptotically stable. $\square$

Finally, we give an example to show the application of the above result.

Example. Consider the scalar impulsive differential equation

$$x'(t) = a(t)x(t) + \int_{-\infty}^{t} b(t, s-t, x(s))\,\mathrm{d}s, \quad t \geqslant 0, \tag{2.7}$$

$$x(t_k) = J_k(x(t_k^-)), \quad k \in Z^+ \tag{2.8}$$

where $a(t) \in C(R^+, R)$, $b(t,u,v)$ is continuous on $R^+ \times (-\infty, 0] \times R$, and $|J_k(x)| \leqslant |1 + c_k||x|$, $k = 1, 2, \ldots$, for $x \in R$ and $\sum_{k=1}^{\infty} |c_k| < \infty$. Suppose $|b(t,u,v)| \leqslant m(u)|v|$, $t \geqslant 0$, and there is a number $A > 0$ such that

$$M \int_{-\infty}^{0} m(u)\,\mathrm{d}u < A \leqslant -a(t), \tag{2.9}$$

where $M = \prod_{k=1}^{\infty}(1 + 2|c_k| + c_k^2)$. Then the zero solution of (2.7) and (2.8) is uniformly asymptotically stable.

In fact, from (2.9) one can choose a constant $L > 0$ and a continuous function $q : (0,\infty) \to (0,\infty)$, q is nonincreasing, such that

$$a(t) + \sqrt{M}\int_{-\infty}^{0} m(u)\,\mathrm{d}u \leqslant -L,$$

$$2\int_{-\infty}^{-q(s)} m(u)\,\mathrm{d}u \leqslant L\sqrt{s}.$$

Let $V(t,x) = V(x) = x^2$,
then

$$V(t_k, J_k(x)) = (J_k(x))^2 \leqslant (1 + 2|c_k| + c_k^2)x^2 = (1 + b_k)V(t_k^-, x),$$

where $b_k = 2|c_k| + c_k^2$. By Remark 2.2, one can easily prove that the zero solution of (2.7) and (2.8) is US. Therefore, without loss of generality, one may assume that $||x||^{[-\infty,t]} \leqslant 1$. For any solution $x(t)$ of (2.7) and (2.8), we have

$$\begin{aligned}
V'(t,x(t)) &\leqslant 2a(t)x^2(t) + 2|x(t)|\int_{-\infty}^{t-q(V(t,x(t)))} m(s-t)|x(s)|\,\mathrm{d}s \\
&\quad + 2|x(t)|\int_{t-q(V(t,x(t)))}^{t} m(s-t)|x(s)|\,\mathrm{d}s \\
&\leqslant 2a(t)x^2(t) + 2|x(t)|\int_{-\infty}^{-q(V(t,x(t)))} m(u)\,\mathrm{d}u \\
&\quad + 2|x(t)| \sup_{\max\{-\infty, t-q(V(t,x(t)))\} \leqslant s \leqslant t} |x(t)| \int_{-\infty}^{0} m(u)\,\mathrm{d}u \\
&\leqslant -Lx^2(t) - 2|x(t)|(\sqrt{M}|x(t)| - \sup|x(s)|)\int_{-\infty}^{0} m(u)\,\mathrm{d}u \\
&= G\left(V(t,x(t)), \sup_{\max\{-\infty, t-q(V(t,x(t)))\} \leqslant s \leqslant t} V(s,x(s))\right),
\end{aligned}$$

where $G(V,U) = -LV - 2\sqrt{V}(\sqrt{MV} - \sqrt{U})\int_{-\infty}^{0} m(u)\,\mathrm{d}u$. Clearly, G is continuous and $G(V,MV) = -LV$. By Theorem 2.4, the zero solution of (2.7) and (2.8) is uniformly and asymptotically stable.

References

[1] A.V. Anokhin, On linear impulsive systems for functional differential equations, Soviet Math. Dokl. 33 (1986) 220–223.

[2] A.V. Anokhin, L. Berezansky, E. Braverman, Exponential stability of linear delay impulsive differential equations, J. Math. Anal. Appl. 193 (1995) 923–941.

[3] D.D. Bainov, G. Kulev, I. Stamova, Global stability of the solutions of impulsive differential difference equations, SUT J. Math. 31 (1995) 55–71.

[4] D.D. Bainov, P.S. Simeonov, Systems with Impulsive Effect: Stability Theory and Applications, Ellis Horwood, Chichester, UK, 1989.

[5] T.A. Burton, Z. Shunian, Unified boundedness, periodicity and stability in ordinary and functional differential equations, Ann. Mat. Pura. Appl. 145 (1986) 129–158.
[6] R.D. Driver, Existence and stability of solutions of a delay-differential system, Arch. Rational Mech. Anal. 10 (1962) 401–426.
[7] K. Gopalsamy, B.G. Zhang, On delay differential equations with impulses, J. Math. Anal. Appl. 139 (1989) 110–122.
[8] R. Grimmer, G. Seifert, Stability properties of Volterra integro-differential equations, J. Differential Equations 19 (1975) 142–166.
[9] J.R. Haddock, J. Terjeki, Liapunov–Razumikhin functions and an invariance principle for functional differential equations, J. Differential Equations 48 (1983) 95–122.
[10] J. Kato, Stability problems in functional differential equations with infinite delays, Funkcial. Ekvac. 21 (1978) 63–80.
[11] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.
[12] V.D. Mil'man, A.D. Myshkis, On the stability of motion in the presence of impulses, Siberian Math. J. 1 (1960) 233–237.
[13] M. Rama Mohana Rao, S.K. Srivastava, Stability of Volterra integro-differential equations with impulsive effect, J. Math. Anal. Appl. 163 (1992) 47–59.
[14] G. Seifert, Liapunov–Razumikhin conditions for asymptotic stability in functional differential equations of Volterra type, J. Differential Equations 16 (1974) 45–52.
[15] J. Shen, Existence and uniqueness of solutions for a class of infinite delay functional differential equations with applications to impulsive differential equations, J. Huaihua Teach. Coll. 15 (1996) 45–51 (in Chinese).
[16] J. Shen, J. Yan, Razumikhin type stability theorems for impulsive functional differential equations, Nonlinear Anal. 33 (1998) 519–531.
[17] J. Shen, Z. Luo, X. Liu, Impulsive stabilization of functional differential equations via Liapunov functionals, J. Math. Anal. Appl. 240 (1999) 1–15.
[18] J. Yan, J. Shen, Impulsive stabilization of functional differential equations by Liapunov–Razumikhin functions, Nonlinear Anal. 37 (1999) 245–255.
[19] J. Yan, A. Zhao, Oscillation and stability of linear impulsive delay differential equations, J. Math. Anal. Appl. 227 (1998) 187–194.
[20] J.S. Yu, B.G. Zhang, Stability theorems for delay differential equations with impulses, J. Math. Anal. Appl. 198 (1996) 285–297.